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Numerical Experiments on the MAC Code for a Slow Flow

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Four finite difference schemes are applied to the numerical integration of the velocity field for low Reynolds number flows, within the framework of the MAC method. The numerical solutions are obtained for the square cavity problem at a Reynolds number equal to 10^{-2}. Their comparison shows that the A.D.I. Douglas–Rachford scheme produces the best results.

1. INTRODUCTION

Flows at low Reynolds numbers constitute a class of practical and technological importance.

In some simple situations, analytical solutions for creeping flows can be obtained by using the linearized form of the Navier–Stokes equations [1]. It seems however that a numerical treatment is the only way for solving complex problems concerned with geodynamics, lubricant flow, glass flow at high temperature.

Among the numerical techniques which have been developed in recent years, the Marker-and-Cell (MAC) method [2] has been applied successfully to transient incompressible viscous flows with free surfaces, such as the splashing drop, the sloshing in a tank, the breaking of a water wave on a slanting wall, etc. Nevertheless, as Pracht [3] has pointed out, the MAC method is restricted to flows for which the Reynolds number must be greater than one; this follows from numerical stability considerations on the original equations of the MAC algorithm, which are written in explicit form. An extension of the applicability of the MAC technique to low Reynolds numbers has been devised by Pracht through an implicit iterative scheme. However, this iterative procedure is time-consuming. At this point, one may wonder whether it is possible to integrate the Navier–Stokes equations numerically within the framework of the MAC code by finite difference schemes which differ from the standard explicit and implicit forms while requiring a faster production code than the iterative method.

The aim of this paper is the description and the comparison for a sample problem,

namely, the square cavity problem at a Reynolds number equal to 10^{-2}, of four finite difference schemes which are unconditionally stable for the heat equation. These schemes are also stable for slow flows when the convective terms are negligible with respect to the viscous terms.

In Section 2, the basic equations are presented and some features of the MAC method are recalled.

In Section 3, the finite difference schemes to be tested are described. They are respectively an alternating direction explicit scheme [4, 5, 6], the odd–even point Hopscotch algorithm [7] and two A.D.I. methods, one of which is due to McKee and Mitchell [8] and the other is the well-known Douglas–Rachford scheme.

The fourth section presents the numerical results for the test problem. The last section is devoted to the discussion of those results. It is shown that the Hopscotch algorithm cannot be used with an increasing time step. The A.D.E. scheme, as far as its computational speed is concerned, suffers from numerical dispersion. The McKee–Mitchell method implies an iterative procedure equivalent to the Pracht's method. Finally, the A.D.I. Douglas–Rachford scheme is twice as fast as either of the overrelaxed iterative methods.

2. Basic Equations. The MAC Method

Consider the plane motion of an isothermal, incompressible viscous fluid in a two-dimensional space referred to rectangular coordinates x and y. Let t denote time, and let $u(x, y, t)$, $v(x, y, t)$ represent the velocity components in the x and y directions, respectively.

For an incompressible fluid, the continuity equation has the form,

$$(\partial u/\partial x) + (\partial v/\partial y) = 0. \tag{1}$$

The Navier–Stokes equations can be written as follows,

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{\partial \phi}{\partial x} + \nu \nabla^2 u, \tag{2}$$

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} = -\frac{\partial \phi}{\partial y} + \nu \nabla^2 v, \tag{3}$$

where ϕ is the ratio of the pressure to the (constant) density, ν is the kinematic viscosity and ∇^2 the Laplacian operator. We shall refer to Eq. (1)–(3) as "system I". Note that the convective terms in (2) and (3) are expressed in conservative form.

With the help of (1), the Navier–Stokes equations may also be expressed in a complete conservative form,

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{\partial \phi}{\partial x} + \nu \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right), \tag{4}$$

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} = -\frac{\partial \phi}{\partial y} - \nu \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right). \tag{5}$$

Eqs. (1), (4), and (5) form the "system II".

We consider the numerical integration of systems I and II by finite differences in space and time. The rectangular domain of integration is covered by a network of fixed rectangular cells. Indices i and j indicate the location of cell centers, with i counting the columns in the x direction, and j counting the rows in the y direction. The placing of the variables in the grid is indicated in Fig. 1. A belt of boundary

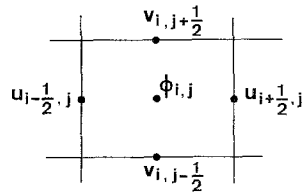

FIG. 1. Field variable placing in a cell.

cells, surrounding the domain of integration, avoids rewriting the finite difference equations for cells adjacent to a boundary and allows for a simple handling of the boundary conditions. The time t is discretized and the current time t is equal to $n\Delta t$, where n is an integer and Δt the time step.

Various methods for integrating the Navier-Stokes equations have been applied to a wide class of fluid flow problems (see, for example, Harlow [9]).

Here, we will concentrate our attention on the basic algorithm proposed in the MAC method [10], which may be characterized by the following features.

1. The velocities at time $t = n\Delta t$ are known from the previous cycle of computation or as initial data.

2. The pressure is obtained in each cell, at time $t = n\Delta t$, by solving the finite difference equation corresponding to the following relation,

$$\nabla^2 \phi = -\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{\partial^2 u^2}{\partial x^2} - 2\frac{\partial^2 uv}{\partial x \partial y} - \frac{\partial^2 v^2}{\partial y^2}, \tag{6}$$

which is obtained from (2) and (3). The finite difference equation is solved by successive overrelaxation. The optimum accelerating factor is determined during the computation itself by a method due to Laloux [11], based on a paper by Carré [12].

3. The finite difference Navier–Stokes equations are used to find the new velocities in the whole mesh. In the original MAC method, these equations are of explicit type.

4. The finite difference form of the left-hand side of (1) is computed. If the absolute numerical values are greater than 10^{-3}, a second iterative procedure proposed by Pracht [13] is introduced for obtaining a higher accuracy.

5. Results are printed. The time is incremented, and the next cycle begins immediately.

The convective terms of the material derivatives in (2) and (3), or (4) and (5) are expressed throughout this paper at time $t = n\Delta t$, and in ZIP form, as it has been suggested by Hirt [14], in order to avoid numerical instability at high Reynolds number. The ZIP form is defined by the following expression,

$$(u^2)^n_{i,j} = u^n_{i-1/2,j} u^n_{i+1/2,j} \,. \tag{7}$$

For the original MAC method, a simple von Neuman stability analysis of the explicit linearized Navier–Stokes equations yields the condition,

$$2\nu\Delta t < \Delta x^2 \Delta y^2/(\Delta x^2 + \Delta y^2), \tag{8}$$

where Δx and Δy are the mesh sizes. The stability requirement (8) can be related to the Reynolds number [3] and yields

$$Re > 1. \tag{9}$$

It is therefore evident that the MAC method is inappropriate for solving low Reynolds number flows. To overcome this difficulty, Pracht [3] has devised an implicit scheme for the full Navier–Stokes equations. This method solves steps 2 and 3 in a single iterative process and is time-consuming due to the slow convergence of the first cycles of computation, particularly when one uses a Gauss–Seidel relaxation method.

The aim of the present paper is to apply various known numerical methods for the integration of parabolic equations to the calculation of step 3 in the MAC algorithm. In the next section, we will successively consider an alternating direction explicit (A.D.E.) method, the "Hopscotch" algorithm, and two A.D.I. methods.

3. Integration of Step 3

In view of the fact that we limit ourselves to small values of the Reynolds number, the diffusion terms in system I and II will be dominant with respect to the convective terms. In order to obtain in step 3 of the computational procedure the values of the velocity components at time $(n+1)\Delta t$, we are therefore allowed to express the convective terms at time $n\Delta t$.

The finite difference schemes to be tested are applied to the second order partial differential operators of the Navier–Stokes equations. For the sake of simplicity, the four schemes are presented below for the simple diffusion equation

$$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right). \tag{10}$$

Let us define the following notation. If $u(x, y, t)$ denotes the solution of the differential equation, $u^n_{i,j}$ represents the solution of the corresponding finite difference equation at the mesh point $x = i\Delta x$, $y = j\Delta y$, and at time $t = n\Delta t$. Furthermore, we set,

$$\begin{aligned} \delta_x{}^2 u^n_{i,j} &= u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}\,, \\ \delta_y{}^2 u^n_{i,j} &= u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}\,. \end{aligned} \tag{11}$$

We now describe briefly the four schemes, checking out the points of interest for our purpose. The reader who wants more details is referred to the original papers, listed in the bibliography.

A.D.E. Method

Larkin [4], Saul'yev [5] and Barakat–Clark [6] have independently developed an alternating direction explicit (A.D.E.) scheme.

During the first sweep of the mesh, from left to right, and from bottom to top, one solves the following equation,

$$\begin{aligned} \frac{u^{n+1}_{1i,j} - u^n_{i,j}}{\Delta t} = \nu\Bigg[&\frac{u^n_{i+1,j} - u^n_{i,j} - u^{n+1}_{1i,j} + u^{n+1}_{1i-1,j}}{\Delta x^2} \\ &+ \frac{u^n_{i,j+1} - u^n_{i,j} - u^{n+1}_{1i,j} + u^{n+1}_{1i,j-1}}{\Delta y^2}\Bigg], \end{aligned} \tag{12}$$

in terms of the auxiliary variable u_1.

A second sweep, from top to bottom and from sight to left, allows for computing the auxiliary variable u_2,

$$\frac{u_{2i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \nu \left[\frac{u_{2i+1,j}^{n+1} - u_{2i,j}^{n+1} - u_{i,j}^{n} + u_{i-1,j}^{n}}{\Delta x^2} + \frac{u_{2i,j+1}^{n+1} - u_{2i,j}^{n+1} - u_{i,j}^{n} + u_{i,j-1}^{n}}{\Delta y^2} \right]. \tag{13}$$

The numerical solution of (10) is obtained by forming the average,

$$u_{i,j}^{n+1} = 0.5\,(u_{1i,j}^{n+1} + u_{2i,j}^{n+1}). \tag{14}$$

A simple von Neumann analysis shows that this scheme is unconditionally stable. For a square cell ($\Delta x = \Delta y = h$), its amplification factor μ_h is given by

$$\mu_h = \frac{1 - \dfrac{2\nu\Delta t}{h^2}(1 - e^{ikh})}{1 + \dfrac{2\nu\Delta t}{h^2}(1 - e^{-ikh})} \tag{15}$$

where k is the wave number.
The local accuracy of the scheme is almost $O(\Delta t^2 + \Delta x^2 + \Delta y^2)$ since the leading terms of the truncation error of Eqs. (12) and (13) are of opposite sign and hence tend to cancel each other in the averaging (14).

Hopscotch Method

This algorithm, primarily due to Gordon [15], has been extensively analyzed and improved by Gourlay [7].

During a first sweep of the network, at mesh points where the sum $n + i + j$ is even, Eq. (10) is solved by the explicit relation,

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \nu \left(\frac{1}{\Delta x^2}\,\delta_x{}^2 + \frac{1}{\Delta y^2}\,\delta_y{}^2 \right) u_{i,j}^{n}, \tag{16}$$

whereas, during a second sweep, at grid points where $n + i + j$ is odd, one uses the implicit relation,

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \nu \left(\frac{1}{\Delta x^2}\,\delta_x{}^2 + \frac{1}{\Delta y^2}\,\delta_y{}^2 \right) u_{i,j}^{n+1}. \tag{17}$$

When the $\delta_x{}^2$ and $\delta_y{}^2$ operators in (16) and (17) are given by (11), they involve their nearest neighbors. In this case, Eq. (17) is of explicit type.

This method can be considered as a new process of splitting-up an A.D.I. scheme of the Peaceman–Rachford type. The algorithm always leads to a two-step process, whatever the number of space dimensions. It is locally (but not globally) equivalent to the Dufort–Frankel scheme. Gourlay has proved that the method is stable provided that the ratio $\Delta t/\Delta x^2$ remains constant.

MKM Method

At first conceived by McKee and Mitchell [8] for partial differential equations with mixed derivatives this scheme can be applied to Eq. (10) as follows.

$$\left[1 + \left(\frac{1}{f} - \frac{1}{2}\frac{\nu\Delta t}{\Delta x^2}\right)\delta_x^2\right] u_{i,j}^*$$

$$= \left[1 + \left(\frac{1}{f} + \frac{1}{2}\frac{\nu\Delta t}{\Delta x^2}\right)\delta_x^2 + \frac{\nu\Delta t}{\Delta y^2}\delta_y^2 + \frac{\nu\Delta t}{f}\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)\delta_x^2\delta_y^2\right] u_{i,j}^n , \quad (18)$$

$$\left[1 + \left(\frac{1}{f} - \frac{1}{2}\frac{\nu\Delta t}{\Delta y^2}\right)\delta_y^2\right] u_{i,j}^{n+1} = u_{i,j}^* + \left(\frac{1}{f} - \frac{1}{2}\frac{\nu\Delta t}{\Delta y^2}\right)\delta_y^2 u_{i,j}^n , \quad (19)$$

where u^* is an intermediate value of u^{n+1}.

For $f < 0$ or $f \geqslant 4$, McKee and Mitchell have shown that this scheme is unconditionally stable. Note that for $f = \infty$, the scheme reduces to the Peaceman–Rachford one, which has been used by Chorin [16] for the numerical integration of the Navier–Stokes equations in a twodimensional space.

D. R. Method

Finally, we will consider the well-known unconditionally stable Douglas–Rachford (D.R) scheme, which can be considered as a perturbation of the backward implicit method. Its split form is given by the following relationships.

$$\left(1 - \frac{\nu\Delta t}{\Delta x^2}\delta_x^2\right) u_{i,j}^* = \left(1 + \frac{\nu\Delta t}{\Delta y^2}\delta_y^2\right) u_{i,j}^n , \quad (20)$$

$$\left(1 - \frac{\nu\Delta t}{\Delta y^2}\delta_y^2\right) u_{i,j}^{n+1} = u_{i,j}^* - \frac{\nu\Delta t}{\Delta y^2}\delta_y^2 u_{i,j}^n . \quad (21)$$

For equal spatial sizes, the amplification factor is,

$$\mu_h = \frac{1 + 16\left(\frac{\nu\Delta t}{h^2}\right)^2 \sin^4\frac{kh}{2}}{\left(1 + 4\frac{\nu\Delta t}{h^2}\sin^2\frac{kh}{2}\right)^2} . \quad (22)$$

4. A Test Problem: The Square Cavity

An isothermal incompressible fluid is contained in a square cavity, with a unit side length (Fig. 2). The fluid is motionless for $t \leqslant 0$. At time $t = 0$, the upper side is instantaneously set in motion in its own plane at constant unit velocity. A vanishing reference pressure is fixed at the middle of the bottom side.

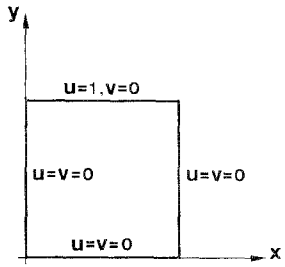

Fig. 2. Square cavity model.

The flow induced by the plate creates a circulatory motion. The fluid moves around a point, called the vortex center, where the velocity vector is zero. Small countervortices develop in both lower corners. For more extensive details on the square cavity problem, see Refs. [17, 18].

In this paper, we study the transient flow for a Reynolds number equal to 10^{-2}, by making use of the MAC algorithm with the four proposed schemes for step 3.

All the computations have been performed on a I.B.M. computer 370/155, in Fortran IV, *G* level, double precision, for a 20×20 network.

The initial dimensionless time step Δt is equal to $2 \cdot 10^{-5}$. At cycles 11, 20, 29, 30, one takes $\Delta t = t$. This leads to time steps respectively equal to $2 \cdot 10^{-4}$, $2 \cdot 10^{-3}$, 0.02, 0.04. In this way, a good description of the transient flow is obtained and the practical numerical stability of the schemes may be tested. The computation with the last time step (0.04) continues till the time 1.68 is reached.

The convergence criterion for the pressure equation is written as,

$$\left| \frac{|\phi_{i,j}^{k}| - |\phi_{i,j}^{k+1}|}{|\phi_{i,j}^{k}| + |\phi_{i,j}^{k+1}|} \right| < \epsilon_1 , \tag{23}$$

where k is an iteration counter. The numerical value of ϵ_1 for all the computations has been taken equal to $2 \cdot 10^{-4}$. The convergence criterion of the second iterative process in step 4 has the same form as Eq. (23) and ϵ takes the value $2 \cdot 10^{-5}$.

The M.K.M. and D.R. schemes integrate system I, whereas the A.D.E. and Hopscotch algorithms deal with system II.

Fairly good results were produced by the Hopscotch method during the ten first cycles of computation. Unfortunately, after the first increase of the time step, the method diverges in two cycles, and horizontal velocity components greater than one appear. This is due to the fact that unconditional stability is only implied by ratios $\Delta t/\Delta x^2$ constant. However, we have to mention that better and stable results were obtained by the "Line Hopscotch" algorithm (cf. [19] for further details on the Line Hopscotch process).

The M.K.M. scheme presented numerical oscillations from cycle 11 onwards. Table I shows the evolution of the u-velocity component on the symmetry axis of

TABLE I

Evolution of the Horizontal Velocity Component Near the Moving Plate for the *MKM* Scheme

Cycle	$F = 12$	$F = -4$	$F = \infty$
11	0.85540	0.85191	0.85596
12	0.84419	0.83856	0.84417
13	0.86232	0.86096	0.86217
14	0.83972	0.83239	0.84067

the cavity at a distance $\Delta y/2$ under the moving plate, for $f = -4$, 12, and ∞. One verifies that the scheme slowly diverges and oscillates around the value 0.85313, obtained by Greenspan [20] for the steady-state problem at zero Reynolds number.

Complete solutions for the transient flow till the steady state have been carried out by the A.D.E., D.R. and Pracht iterative methods. Table II compares the

TABLE II

Comparison of the Differences Between the Numerical Solutions by the Proposed Scheme and That by the Greenspan Method

Method	ADE	PRACHT	DR
	$0(10^{-2} - 10^{-3})$	$0(10^{-3} - 10^{-4})$	$0(10^{-4} - 10^{-5})$

differences between the results given by these schemes and those by the Greenspan method, cited above, for the steady-state horizontal velocity component on the symmetry axis. It is established that the D.R values are close to those by Greenspan.

Table III gives the C.P.U. times for a complete execution (70 cycles) of

TABLE III
CPU Times for Various Methods

Method	ADE	D.R.	PRACHT	
			Gauss–Seidel	Overrelaxed
CPU time	19 min 34 sec	12 min 19 sec	41 min 58 sec	24 min 11 sec

the problem by these various algorithms. One can see that the computational speed of the D.R. scheme is twice that of Pracht's overrelaxed method. The time saving is a consequence of the direct inversion of the algebraic system.

5. Discussion

The amplification factor μ of Eq. (10) is usually defined as the amplitude ratio of a harmonic wave $a(t)\exp((-1)^{1/2}\, k \cdot x)$ at time $t + \Delta t$ and at time t. ($k \cdot x$ is a scalar product). By (10), one finds,

$$\mu = \exp[-\nu(k_x^2 + k_y^2)\, t], \tag{24}$$

where k_x and k_y denote the wave numbers in the x and y directions, respectively. They are related to the corresponding wave length λ_x and λ_y by the expressions $k_x = 2\pi/\lambda_x$ and $k_y = 2\pi/\lambda_y$.

We will suppose, for the sake of clarity, that $k_x = k_y = k$ and $\Delta x = \Delta y = h$. By replacing t by one time step Δt, Eq. (24) becomes,

$$\mu = \exp[-(2\nu \Delta t/h^2)(kh)^2]. \tag{25}$$

As we have denoted the amplification factor of the corresponding finite difference equations by μ_h, we may define the numerical dissipation α and the numerical dispersion β of a wave as,

$$\alpha = |\mu| - |\mu_h|, \qquad \beta = \arg(\mu/\mu_h). \tag{26}$$

By (25) and (26), in the case of a pure diffusion equation, the finite difference scheme must be such that its amplification factor is real, hence achieving a vanishing numerical dispersion. It should also be as close as possible to the value given by Eq. (25) in order to minimize the numerical dissipation.

By (15) and (22), one sees that the A.D.E. scheme involves a nonvanishing numerical dispersion, whereas the D.R. scheme is free from this shortcoming.

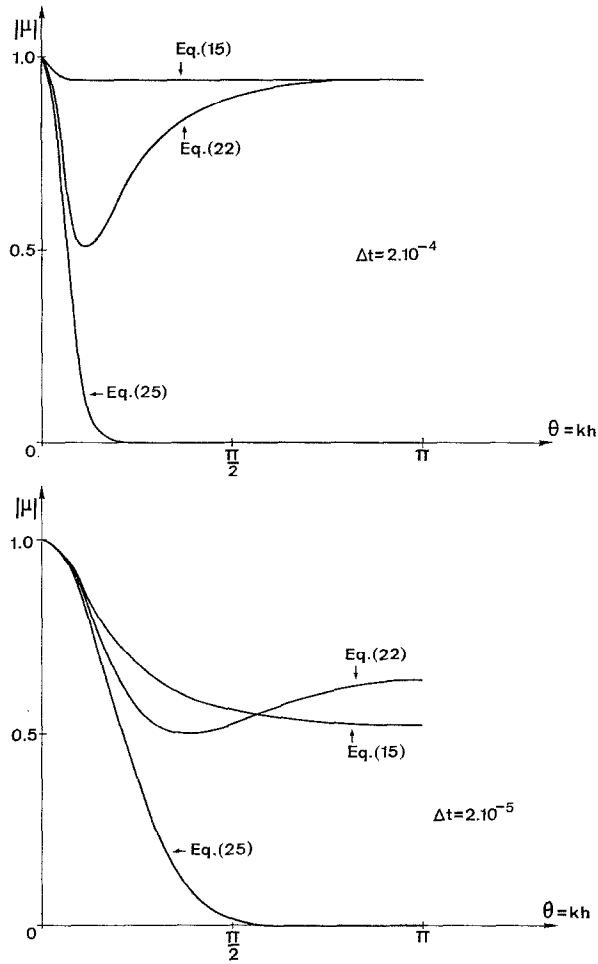

FIG. 3. Amplification factor modulus versus phase angle.

Figure 3 displays the evolution of the amplification factor moduli of (15), (22) and (25) versus the phase angle $\theta = kh$, for time steps equal to $2 \cdot 10^{-5}$ and $2 \cdot 10^{-4}$.

It appears that the amplification factor moduli for (15) and (22) are rather far away from that of the exact solution (25). Particularly, the A.D.E. scheme does not present any damping of the perturbations generated either by round-off error or by a brutal increase of the time step. Indeed, for $\Delta t = 2 \cdot 10^{-4}$, its modulus remains near one. This phenomenon, conjugated with numerical dispersion, leads to worse $D_{i,j}$ for this scheme than for the D.R. one. As a result, the correction iterative procedure on the continuity equation converges more slowly, and the C.P.U. time

is therefore about 1.6 times longer than for the D.R. method. Furthermore, extension of this method to free surface flows is not a trivial task.

The oscillatory behavior of the M.K.M. scheme can be explained in the following way. The numerical scheme (18), (19), retains an order of accuracy $O(\Delta t^2)$, but, as we work with pressure gradients expressed at the old time level, this induces a loss of accuracy inconsistent with the local accuracy of the scheme. The discretization error remains at the $O(\Delta t)$ level, and it is particularly sensitive near the boundaries of the flow region. To avoid this defect, we have to integrate iteratively step 2 and 3, as has been proposed by Pracht, thus obtaining a slow production code.

However, the f parameter introduced by McKee and Mitchell can be used in order to minimize the numerical dissipation. One knows that the $\lambda = 2\Delta x$ waves ($\theta = \pi$) are the finest ones that the network can support. In case of any perturbation of the numerical field, they grow at the fastest rate. One thus may adjust the f parameter value in such a way that for $\theta = \pi$, the amplification factor modulus of the M.K.M. finite difference scheme is the same as for the exact solution (25), namely $\mu = \exp[-2\nu\Delta t\theta^2/h^2]$. One obtains a curve for μ_h lying very close to that given by (25). Nevertheless, one has to be careful that this f value does not involve an undue growth of the principal part of the truncation error, and respects the stability requirements.

We have seen that the D.R. scheme is a good one. Its accuracy is convenient; its computational speed is twice that of Pracht's method. Its stability for an increase of time step is remarkable.

In conclusion, we have adopted the application of the Douglas–Rachford scheme for the numerical integration of the Navier–Stokes equations at low Reynolds numbers, within the framework of the MAC method, because it is accurate, fast, and easily programmed.

We will consider its application to viscous free surface flows in a forthcoming paper.

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